REALIZING DEFORMATIONS OF CURVES USING LUBIN–TATE FORMAL GROUPS

BY

Barry Green*

Department of Mathematics, University of Stellenbosch
7602 Stellenbosch, South Africa
e-mail: bwg@sun.ac.za

ABSTRACT

Let k be an algebraically closed field of characteristic $p > 0$ and R be a suitable valuation ring of characteristic 0, dominating the Witt vectors $W(k)$. We show how Lubin–Tate formal groups can be used to lift those order p^n automorphisms of $k[\![Z]\!]$ to $R[\![Z]\!]$, which occur as endomorphisms of a formal group over k of suitable height. We apply this result to prove the existence of smooth liftings of galois covers of smooth curves from characteristic p to characteristic 0, provided the p-part of the inertia groups acting on the completion of the local rings at the points of the cover over k are p-power cyclic and determined by an endomorphism of a suitable formal group over k.

1. Introduction

The aim of this note is to build on recent work on liftings of galois covers of smooth curves defined over an algebraically closed field k of characteristic p, to relative smooth galois covers of curves over a suitable valuation ring R of characteristic 0, dominating the Witt vectors $W(k)$.

Due to a local-global-principle for lifting galois covers of smooth curves, this problem reduces to the problem of lifting groups of automorphisms of a formal power series ring over k to groups of automorphisms of a formal power series

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ring over the valuation ring R dominating $W(k)$. Thus we are led to the study of finite order automorphisms of the p-adic open disc and the geometry of their fixed points. In this context those of p-power order are crucial; see, for example, [G-M1], [G-M2] and [G]. The local-global-principle for liftings gives necessary and sufficient conditions, whereby liftings of the inertia groups acting on the completions of the local rings at the points of a galois cover of smooth curves over k, to smooth galois covers of the p-adic open disc over R, ensures a global lifting to a galois covering of smooth relative curves over R. The completed local rings are formal power series rings over k and R respectively, and so provide the setting for this investigation.

In this short note we show how Lubin–Tate formal groups can be used to lift those order p^n automorphisms of $k[\![Z]\!]$ which occur as endomorphisms of a formal group $\overline{F}$ over k of suitable height h. This result is proved in Theorem 4.1 and follows directly from a number of classical results on formal groups, which we recall in section 3 for the convenience of the reader.

In the final paragraph of section 4 we apply Theorem 4.1 together with the local-global-principle mentioned above to prove the existence of smooth liftings of galois covers of smooth curves from characteristic p to characteristic 0, provided the p-part of the inertia groups acting on the completion of the local rings at the points of the cover over k are p-power cyclic and determined by an endomorphism of a suitable formal group over k. Precisely we prove:

THEOREM 4.2: *Let $f\colon C \longrightarrow C/G := D$ be a G-galois cover of proper integral smooth curves over k. For each $y \in C$ we denote the p-part of the corresponding inertia group by $(I_y)_p$. Suppose that $(I_y)_p$ is p^{n_y}-cyclic and that, if $n_y \geq 3$, the embedding $(I_y)_p \subset \mathrm{Aut}_k(\hat{\mathcal{O}}_{C,y}) \cong \mathrm{Aut}_k(k[\![Z]\!])$ factors through $\mathrm{Aut}_k(\overline{F}(Z_1, Z_2))$ for some one dimensional formal group $\overline{F}(Z_1, Z_2)$ over k of height*

$$h_y = rp^{n_y - 1}(p-1), \quad (r,p) = 1.$$

Let $n = \max_y(n_y)$ and $\zeta_{(n)}$ be a primitive p^n-root of unity. Then f can be lifted over $R = W(k)[\zeta_{(n)}]$ as a G-galois cover of smooth integral proper R-curves.

2. Motivating questions

SITUATION. Let k be an algebraically closed field of characteristic $p > 0$, and C be a proper smooth integral curve of genus g over k. Let R be a complete discrete valuation ring dominating the ring of Witt vectors $W(k)$ and having uniformizing parameter π. We denote the quotient field of R by K, which is assumed finite over $\mathrm{Quot}(W(k))$, and the corresponding integral and algebraic closures by $\tilde{R}$ and $\tilde{K}$.

MOTIVATING QUESTIONS: *Let G be a finite subgroup of $\mathrm{Aut}_k(C)$ and suppose that $C \longrightarrow D = C/G$ is a finite galois cover of smooth proper integral curves over k.*

1. Global question: *Is it possible to find R as above and a finite galois cover of smooth proper integral relative curves over R, $\mathcal{C} \longrightarrow \mathcal{D} = \mathcal{C}/G$, which lifts the given cover $C \longrightarrow D$?*

2. Local question: *Let $y \in C$ and I_y be the associated inertia group, so that $I_y \subseteq \mathrm{Aut}_k(\hat{\mathcal{O}}_{C,y}) \cong \mathrm{Aut}_k(k[\![Z]\!])$ for a suitable parameter Z. Is it possible to find R as above such that the inertia group lifts to $I_y \subseteq \mathrm{Aut}_R(R[\![Z]\!])$?*

Remark: Clearly if the global question is satisfied, then for each $y \in C$ we have $I_y \subseteq \mathrm{Aut}_R(\hat{\mathcal{O}}_{\mathcal{C},y}) \cong \mathrm{Aut}_R(R[\![Z]\!])$ for a suitable parameter Z, so the local question is satisfied for each point $y \in C$. The converse of this result is also true and is the local-global-principle for liftings referred to in the introduction. We refer the reader to [G-M1], Section III, for details and to [B-Mé] and [He] for alternative proofs.

HISTORICAL BACKGROUND. For a detailed account on developments related to this problem we refer to [Se-O], [O2], [O-Se-S], [Se-S2], and [G-M1]. Briefly, we note that if $(|G|, p) = 1$ the answer to the global question is yes for any R, by Grothendieck, SGA I. If $|G| > 84(g-1)$ then the answer is no, due to a contradiction using Hurwitz bounds. In chararacteristic p there exist curves C/k such that one can choose G with $|G| > 84(g-1)$, see [Ro], but in characteristic 0 the order of the automorphism group of a curve of genus g is at most $84(g-1)$. One remarks that if G is abelian then, by Nakajima, [N], the bounds for $G \subset \mathrm{Aut}_k(C)$ are the same in any characteristic and so in this case one doesn't expect a contradiction using bounds. So one speculates that for such G smooth liftings may always exist, and the first case one studies is for G cyclic. The first step in this direction, proved by Oort–Sekiguchi–Suwa in [O-Se-S], shows that if G is cyclic of order pe, with $(e, p) = 1$, the answer is yes if R contains a primitive p-root of unity, say ζ.

Following these results it became natural to conjecture the following generalisation (see [Se-O], [O1] and [O2]):

OORT–SEKIGUCHI CONJECTURE: *The answer to the global lifting question is positive if G is a cyclic group.*

In a number of recent papers (see, for example, [G-M1], [B], [M], [P] and [G]) necessary conditions for the solvability of the lifting problem when the p-parts of the inertia groups aren't cyclic are given. Examples of elementary p abelian covers which cannot be lifted over any R are also given. Concerning the conjecture, one of the main results of [G-M1] answers it positively for G-galois covers whose inertia groups are $p^n e$-cyclic with $n \leq 2$ and $(e,p) = 1$. More precisely one has:

THEOREM 2.1 ([G-M1]): *Let $f\colon C \longrightarrow C/G := D$ be a G-galois cover of proper smooth curves over k. Assume that the inertia groups are $p^n e$-cyclic with $n \leq 2$ and $(e,p) = 1$. Then f can be lifted over $R = W(k)[\zeta_{(2)}]$ as a G-galois cover of smooth R-curves, where $\zeta_{(2)}$ is a primitive p^2-root of unity.*

By the local-global-principle for liftings the crucial question is the local one, namely that of the existence of liftings of G-galois covers of formal power series rings $k[\![Z]\!]/k[\![Z]\!]^G = k[\![T]\!]$ over k (here $T = \prod_{\sigma \in G} Z^\sigma$), to G-galois covers of the formal power series rings $R[\![Z]\!]/R[\![Z]\!]^G = R[\![T]\!]$ over R, for R and G as above. This is the condition which ensures smoothness of the lifting of curves.

The formal power series ring $R[\![Z]\!]$ is a two dimensional local ring with maximal ideal generated by Z and π, and its height one prime ideals are all principal. If $\sigma \in \mathrm{Aut}_R(R[\![Z]\!])$ then we can write

$$\sigma(Z) = a_0 + a_1 Z + a_2 Z^2 + a_3 Z^3 + \cdots,$$

where $a_0 \in \pi R$ and $a_1 \in R^\times$. Observe that reducing the coefficients of σ modulo (π), we obtain a canonical surjective group homomorphism

$$\Psi\colon \mathrm{Aut}_R(R[\![Z]\!]) \longrightarrow \mathrm{Aut}_k(k[\![Z]\!]).$$

In view of the discussion above we are interested in the following questions:

LIFTING QUESTIONS: 1. *If $\overline{\sigma}$ is a k-automorphism of $k[\![Z]\!]$ of finite order, when can we find a lifting of σ to an R-automorphism of $R[\![Z]\!]$ of the the same order, i.e., an R-automorphism σ of $R[\![Z]\!]$ of the the same order such that $\Psi(\sigma) = \overline{\sigma}$?*

More generally, if $\overline{G}$ is a finite subgroup of $\mathrm{Aut}_k(k[\![Z]\!])$, when can one find a section of $\overline{G}$ in $\mathrm{Aut}_R(R[\![Z]\!])$ with respect to Ψ?

2. *If G is a finite group, give criteria depending on the structure of the group which need to be satisfied in order that a realisation $G \subset \mathrm{Aut}_k(k[\![Z]\!])$ can be lifted to $G \subset \mathrm{Aut}_R(R[\![Z]\!])$, for some complete discrete valuation ring R dominating $W(k)$?*

The first question is the one we are concerned with here, for those order p^n automorphisms $\overline{\sigma}$, which are also automorphisms of a formal group over k of suitable height.

3. Lubin–Tate formal group prerequisites

In this section, for the convenience of the reader we briefly recall the classical results on formal groups over fields of characteristic p and Lubin–Tate liftings to formal groups over the appropriate valuation ring extending the Witt vectors. These results were obtained by Hasse [H], Lazard [Laz], Tate and Lubin [L-T]. As a general reference we mention the book by Hazewinkel, [Haz]: Precisely, Theorem 19.4.1, on page 170 and Theorem 20.2.13, on pages 183–187 for the classical results 3.1 and 3.2, and section 8.1 on pages 43–45 and Proposition 18.3.11 on page 152 for the classical results 3.3.

Assume k is a separably closed field of characteristic p and let $\overline{F}$ be a formal group defined over k. The ring of $\overline{F}$ endomorphisms,

$$\mathrm{End}_k(\overline{F}(Z_1, Z_2)) \subset k[\![Z]\!],$$

is defined as follows: $\alpha(Z) \in \mathrm{End}_k(\overline{F}(Z_1, Z_2))$ if $\alpha(0) = 0$ and $\alpha(\overline{F}(Z_1, Z_2)) = \overline{F}(\alpha(Z_1), \alpha(Z_2))$. For the ring structure on $\mathrm{End}_k(\overline{F}(Z_1, Z_2))$, addition is determined by the formal group, i.e., $(\alpha_1 +_{\overline{F}} \alpha_2)(Z) = \overline{F}(\alpha_1(Z), \alpha_2(Z))$, and multiplication given by the usual composition of series. Note that with respect to multiplication (the usual composition law) we have $\mathrm{Aut}_k(\overline{F}(Z_1, Z_2)) \subset \mathrm{Aut}_k(k[\![Z]\!])$.

For $\alpha(Z) \in \mathrm{End}_k(\overline{F}(Z_1, Z_2))$ the height of α is defined by $\mathrm{ht}(\alpha) = 0$ if $\alpha(Z) = 0$, and $\mathrm{ht}(\alpha) = r$ if $q = p^r$ is the highest power such that $\alpha(Z) = \beta(Z^q)$, where $\beta(Z^q) \in k[\![Z^q]\!]$. The height of $\overline{F}$ is defined to be

$$\mathrm{ht}(\overline{F}) = \mathrm{ht}([p]_{\overline{F}}),$$

where

$$[p]_{\overline{F}} = Z +_{\overline{F}} Z +_{\overline{F}} \cdots +_{\overline{F}} Z,$$

addition p-fold. Having introduced the above notations we recall the following classical results we need:

Classical Results 3.1: *Over a separably closed field of characteristic p the 1 dimensional formal groups $\overline{F}$ are classified up to isomorphism by their heights. Further, if $\mathrm{ht}(\overline{F}) = h$ and $q = p^h$, then*

(i) *$\overline{F}$ is defined over $\mathbb{F}_q$ and $E_h := \mathrm{End}_{\mathbb{F}_q}(\overline{F}(Z_1, Z_2)) = \mathrm{End}_k(\overline{F}(Z_1, Z_2))$;*

(ii) *E_h is a free module of rank h^2 over $\mathbb{Z}_p$;*

(iii) *$D_h := E_h \otimes \mathbb{Q}_p$ is a division algebra of rank h^2 over $\mathbb{Q}_p$;*

(iv) *E_h is the maximal order of D_h;*

(v) *the center of D_h is $\mathbb{Q}_p$.*

CLASSICAL RESULTS 3.2: *In the above situation of the division algebra D_h over $\mathbb{Q}_p$:*

(i) *There is a unique exponential valuation, say v, extending that on $\mathbb{Q}_p$ to D_h.*

(ii) $E_h = \{x \in D_h : v(x) \geq 0\}$.

(iii) *The Skolem Noether Theorem: For any subfields L, L' of D_h, if $L \cong L'$ over $\mathbb{Q}_p$ then this isomorphism extends to an inner automorphism of D_h.*

(iv) *Any extension of $\mathbb{Q}_p$ of degree h, say L, can be embedded in D_h as a maximal subfield.*

(v) *Let e denote the ramification index of D_h over $\mathbb{Q}_p$ and f the residue class degree. Then $e = f = h$.*

(vi) *If D_h contains a primitive p^n-root of unity, then $p^{n-1}(p-1)|h$.*

CLASSICAL RESULTS 3.3: *Let $R = W(\mathbb{F}_q)[\zeta_{(n)}]$ with parameter π, where $q = p^h$, $h = rp^{n-1}(p-1)$ with $(r,p) = 1$ and $\zeta_{(n)}$ is a primitive p^n-root of unity. Then $F_l(Z_1, Z_2) := f_l^{-1}(f_l(Z_1) + f_l(Z_2))$, where $f_l(Z) := Z + \pi^{-1} f_l(Z^{p^r})$ defines a Lubin–Tate formal group such that:*

(i) *the map $\varphi \colon R \longrightarrow \mathrm{End}_R(F_l(Z_1, Z_2))$ defined by*

$$\varphi(a) = [a]_{F_l}(Z) = f_l^{-1}(a f_l(Z))$$

is an injective ring homomorphism;

(ii) $[a]_{F_l}(Z) \equiv aZ \bmod Z^2$ *and* $[\pi]_{F_l}(Z) \equiv Z^{p^r} \bmod \pi$;

(iii) *$\overline{F_l}(Z_1, Z_2)$, obtained by reducing the coefficients of $F_l(Z_1, Z_2) \bmod \pi$, is a formal group over $\mathbb{F}_q$ of height h;*

(iv) *the map $v \colon \mathrm{End}_R(F_l(Z_1, Z_2)) \longrightarrow \mathrm{End}_{\mathbb{F}_q}(\overline{F_l}(Z_1, Z_2))$, obtained by reducing coefficients modulo π, is an injective ring homomorphism.*

4. Lifting order p^n formal group automorphisms

In this section we prove the following theorem:

THEOREM 4.1: *Let k be a separably closed field of characteristic $p > 0$ and n be a positive integer. Suppose G is an order p^n cyclic group and that in the diagram below the morphism i factors through $\mathrm{Aut}_k(\overline{F}(Z_1, Z_2))$ for some one dimensional formal group $\overline{F}(Z_1, Z_2)$ over k of height $h = rp^{n-1}(p-1)$, $(r,p) = 1$:*

$$\begin{tikzcd} \mathrm{Aut}_k(k[\![Z]\!]) & \mathrm{Aut}_k(\overline{F}(Z_1, Z_2)) \arrow[l] \\ G \arrow[u, "i"] \arrow[ur] & \end{tikzcd}$$

Then $i(G) \subset \mathrm{Aut}_{\mathbb{F}_q}(\mathbb{F}_q[\![Z]\!]) \subset \mathrm{Aut}_k(k[\![Z]\!])$, where $q = p^h$, and there exists a lifting of $i: G \to \mathrm{Aut}_{\mathbb{F}_q}(\mathbb{F}_q[\![Z]\!])$ over $R = W(\mathbb{F}_q)[\zeta_{(n)}]$, which makes the diagram

$$\begin{tikzcd}
\mathrm{Aut}_R(R[\![Z]\!]) \arrow[r] & \mathrm{Aut}_{\mathbb{F}_q}(\mathbb{F}_q[\![Z]\!]) & \mathrm{Aut}_{\mathbb{F}_q}(\overline{F}(Z_1, Z_2)) \arrow[l] \\
& G \arrow[ul] \arrow[u, "i"] \arrow[ur] &
\end{tikzcd}$$

commutative.

Proof: We first note that using the classical results from section 3 it follows that

$$\mathrm{End}_k(\overline{F}(Z_1, Z_2)) = \mathrm{End}_{\mathbb{F}_q}(\overline{F}(Z_1, Z_2)) \cong \mathrm{End}_{\mathbb{F}_q}(\overline{F_l}(Z_1, Z_2))$$

and that

$$R \overset{\varphi}{\hookrightarrow} \mathrm{End}_R(F_l(Z_1, Z_2)) \overset{\iota}{\hookrightarrow} \mathrm{End}_{\mathbb{F}_q}(\overline{F_l}(Z_1, Z_2)) = E_h.$$

We set $\rho = \iota \circ \varphi$ and observe that it is an injective ring homomorphism. Next we set $\overline{\tau} = \rho(\zeta_{(n)}) \in E_h$, a primitive p^n-root of unity. Note that if $\overline{\sigma} \in E_h$ is any other primitive p^n-root of unity, then in D_h the $\mathbb{Q}_p$-isomorphism

$$\mathbb{Q}_p(\overline{\tau}) \cong \mathbb{Q}_p(\overline{\sigma})$$

extends to an inner automorphism of D_h by the Skolem Noether Theorem (see 3.2(iii)). Hence $\overline{\sigma} = \overline{\gamma}\,\overline{\tau}\,\overline{\gamma}^{-1}$ for some $\overline{\gamma} \in D_h$.

We next recall that if Π is any uniformizing parameter of D_h, then $\mathbb{Q}_p(\Pi)$ is a totally ramified extension of $\mathbb{Q}_p$ of degree h. Now $\mathbb{Q}_p(\overline{\tau}) \subset D_h$ is a totally ramified extension of $\mathbb{Q}_p$ of degree $p^{n-1}(p-1)$. Let L be a totally ramified extension of $\mathbb{Q}_p(\overline{\tau})$ of degree r. Then L can be embedded in D_h over $\mathbb{Q}_p(\overline{\tau})$ as a totally ramified extension of $\mathbb{Q}_p$ of degree h. Let Π be a uniformizing parameter for this embedding, which will then also be a uniformizing parameter for D_h. We have

$$\mathbb{Q}_p \subset \mathbb{Q}_p(\overline{\tau}) \subset \mathbb{Q}_p(\Pi) \subset D_h.$$

Next we show that the element $\overline{\gamma}$ such that $\overline{\sigma} = \overline{\gamma}\,\overline{\tau}\,\overline{\gamma}^{-1}$ can be chosen so that it is a unit in $E_h = \mathrm{End}_{\mathbb{F}_q}(\overline{F_l}(Z_1, Z_2))$. From this it follows that

$$\overline{\gamma} \in \mathrm{Aut}_{\mathbb{F}_q}(\overline{F_l}(Z_1, Z_2)) \subset \mathrm{Aut}_{\mathbb{F}_q}(\mathbb{F}_q[\![Z]\!]).$$

Clearly $\overline{\gamma} = u\Pi^m$ for some unit $u \in E_h$. Hence

$$\overline{\sigma} = u\Pi^m\overline{\tau}\Pi^{-m}u^{-1} = u\overline{\tau}u^{-1},$$

since Π and $\overline{\tau}$ both lie in the field $\mathbb{Q}_p(\Pi)$ so commute past each other. We replace $\overline{\gamma}$ by u.

Finally, let

$$\tau = \varphi(\zeta_{(n)}) = [\zeta_{(n)}]_{F_l}(Z)$$

and choose $\gamma \in \mathrm{Aut}_R(R[\![Z]\!])$ such that $\gamma \bmod \pi = \overline{\gamma}$. Set $\sigma = \gamma\tau\gamma^{-1}$, which is an order p^n automorphism of $R[\![Z]\!]$ lifting $\overline{\sigma}$ as required.

Observations. 1. We note that if $p \neq 2$, then the only finite p-groups contained in the multiplicative group of D_h are the p power cyclic groups. Indeed, if $G \subset D_h^\times$ is a non-cyclic p-group, then as $p \neq 2$, G contains a non-cyclic abelian group, say G_0. But then as $\mathbb{Q}_p$ is the center of D^h (see 3.1(v)) it follows that $\mathbb{Q}_p(G_0)$ is a subfield of D_h. The only finite subgroups of the multiplicative group of a field are the cyclic ones, which is a contradiction as $G_0 \subset \mathbb{Q}_p(G_0)^\times$. Hence for $p \neq 2$ the formal groups can only be applied to the lifting problem for cyclic groups. For $p = 2$ the division algebra D_h contains the quarterion group, which doesn't contain a non-cyclic abelian subgroup — so the argument does not apply here.

2. The order p^n automorphisms of $R[\![Z]\!]$, respectively $k[\![Z]\!]$, which are determined by a Lubin–Tate formal group as above are very special, as can be seen from their fixed point geometry and associated Hurwitz data, which is described briefly in [G-M2], pages 279–281. There, this is discussed as part of a general study of order p automorphisms of the p-adic open disc, their fixed point geometry types and classification under certain assumptions. Results obtained there, as well as those obtained subsequently by Henrio in [He] characterizing possible fixed point geometry types for order p automorphisms, show that the order p automorphisms determined by a Lubin–Tate formal group form a very small class among those of order p of $R[\![Z]\!]$, respectively $k[\![Z]\!]$.

Realizing smooth liftings of galois covers of curves. Applying Theorem 4.1 above together with Theorem 2.1 and the local-global-principle for smooth liftings mentioned in the introduction, we obtain the existence of smooth liftings of galois covers of smooth curves from characteristic p to characteristic 0, provided the p-part of the inertia groups acting on the completion of the local rings at the points of the cover over k are p-power cyclic and determined by the endomorphism of a suitable formal group over k. More precisely:

Theorem 4.2: *Let $f\colon C \longrightarrow C/G := D$ be a G-galois cover of proper integral smooth curves over k. For each $y \in C$ we denote the p-part of the corresponding inertia group by $(I_y)_p$. Suppose that, $(I_y)_p$ is p^{n_y}-cyclic and that, if $n_y \geq 3$, the* *embedding* $(I_y)_p \subset \mathrm{Aut}_k(\hat{O}_{C,y}) \cong \mathrm{Aut}_k(k[\![Z]\!])$ *factors through* $\mathrm{Aut}_k(\overline{F}(Z_1, Z_2))$ *for some one dimensional formal group* $\overline{F}(Z_1, Z_2)$ *over* k *of height*

$$h_y = rp^{n_y - 1}(p-1), \quad (r,p) = 1.$$

Let $n = \max_y(n_y)$ *and* $\zeta_{(n)}$ *be a primitive* p^n*-root of unity. Then* f *can be lifted over* $R = W(k)[\zeta_{(n)}]$ *as a* G*-galois cover of smooth integral proper* R*-curves.*

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